

LECTURE 15

16 February 2024 09:40

$$Q.I. \quad A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\lambda_1 = 2, \quad \lambda_2 = 3, \quad \lambda_3 = 5$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\lambda_1 = 2 \quad (A - \lambda_1 I) X_1 = 0$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \textcircled{1} & 1 & 4 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_2 = K, \quad x_3 = 0$$

$$x_1 + x_2 + 4x_3 = 0 \Rightarrow x_1 = -K$$

$$X_1 = \begin{bmatrix} -K \\ K \\ 0 \end{bmatrix} = K \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_1 = 3 \quad \begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (A - \lambda_2 I) X_2 = 0$$

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ 0 & \textcircled{1} & 4 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = K, \quad x_3 = 0$$

$$x_2 + 4x_3 = 0 \Rightarrow x_2 = 0$$

$$X_2 = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} = K \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 5 \quad \begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (A - \lambda_3 I) X_3 = 0$$

$$\begin{bmatrix} 0 & -3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (K, 3, 1, 3)$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ -2 & 1 & 4 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad x_3 = K \quad -3x_2 + 6x_3 = 0$$

$$x_2 = 2K$$

$$-2x_1 + x_2 + 4x_3 = 0$$

$$x_1 = 3K$$

$$X_2 = \begin{bmatrix} 3K \\ 2K \\ K \end{bmatrix} = K \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Q.2. $A - \lambda I = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$ $\lambda_1 = -3, \lambda_2 = -3, \lambda_3 = 5$

$$(A - \lambda_1 I) X_1 = 0 \quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 2x_2 - 3x_3 = 0$$

$$x_2 = K_1 \quad x_3 = K_2$$

$$X_{12} = \begin{bmatrix} 3K_2 - 2K_1 \\ K_1 \\ K_2 \end{bmatrix} \quad (K_1, K_2) = (1, 0) \quad x_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \quad (K_1, K_2) = (0, 1)$$

$$(A - \lambda_3 I) X_3 = 0 \quad \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & 16 & 32 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \textcircled{1} & 2 & 5 & 0 \\ 0 & \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_3 = k \quad x_2 + 2x_3 = 0 \Rightarrow x_2 = -2k$$

$$x_1 + 2x_2 + 5x_3 = 0 \Rightarrow x_1 = -k$$

$$x_3 = \begin{bmatrix} -k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1/4 & 1/2 & 3/4 \\ 1/8 & 1/4 & 5/8 \\ -1/8 & -1/4 & 3/8 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ -1 & -2 & 3 \end{bmatrix}$$

Eg 1 $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\det(A - \lambda I) = 0$

$$(A - \lambda I) = \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} \quad \det(A - \lambda I) = (1-\lambda)(3-\lambda) \Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}$$

$$A^2 - 4A + 3I = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 0 & 12 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 0 & 12 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Cayley Hamilton theorem: Every square matrix satisfies its charact. eqⁿ.

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_1\lambda + a_0 = 0$$

$$\text{Then } A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \dots + a_1A + a_0I = 0$$

H.W. Calculate $P^{-1}AP = D$ ✓

$$P^{-1}A^{1000}P = D^{1000}$$

$$A^{1000} = PD^{1000}P^{-1}$$

A^{1000}